ON PRIME ENDS AND LOCAL CONNECTIVITY

LASSE REMPE

Dedicated to the memory of Professor Gerald Schmieder

Abstract. Let $U \subset \hat{\mathbb{C}}$ be a simply connected domain whose complement $K = \hat{\mathbb{C}} \setminus U$ contains more than one point. We show that the impression of a prime end of U contains at most two points at which K is locally connected. This is achieved by establishing a characterization of local connectivity of K at a point $z_0 \in \partial U$ in terms of the prime ends of U whose impressions contain z_0 and then invoking a result of Ursell and Young [UY].

1. Introduction

The theory of *prime ends* was developed by Carathéodory [C] in 1913. One of its central theorems states that the complement K of a simply connected domain $U = \hat{\mathbb{C}} \setminus K$, $\#K > 1$, is locally connected if and only if every prime end has trivial impression, which in turn is equivalent to any Riemann map $\varphi : \mathbb{D} \to U$ having a continuous extension to $\partial\mathbb{D}$. (See Section 3 for a short introduction to the standard definitions and terminology of prime end theory.) This note investigates the question whether there is a relationship between local connectivity of K *at a point* $z_0 \in \partial U$ — i.e., the existence of arbitrarily small connected neighborhoods of z_0 in K,[1] compare Section 2 — and the structure of the prime ends of U whose impressions contain z_0.

This question seems very natural, and may be of particular interest due to the prominence that local connectivity of Julia sets and the Mandelbrot set at certain points has received in recent years (see e.g. [H, K]). However, it does not appear to have received any treatment in the literature so far.

A naive hope might be that K is locally connected at z_0 if and only if every prime end impression which contains z_0 is trivial, but this is false, as the well-known case of the "double comb" shows (Figure 1(a)). However, study of this and similar examples suggests that a nontrivial impression should not contain "too many" points of local connectivity. In this note, we demonstrate that "not too many" can be made very precise. In fact, the example in Figure 1(a) is already best possible.

1.1. Theorem (Prime ends and local connectivity).
Let $U \subset \hat{\mathbb{C}}$ be a simply connected domain such that $K := \hat{\mathbb{C}} \setminus U$ contains more than one point, and let p be a prime end of U. Then the impression $I(p)$ contains at most two points at which K is locally connected.

Date July 4, 2008.
2000 *Mathematics Subject Classification.* Primary 30C35; Secondary 54F15, 30D05, 37F10.
Supported in part by EPSRC grant EP/E017886/1.

[1] Sometimes this property is instead referred to as *connected im kleinen*.

Remark. We also show that, if furthermore the prime end p is *symmetric* (see Section 4), then $I(p)$ contains at most *one* point at which K is locally connected.

The proof of Theorem 1.1 uses a result (Theorem 4.4) concerning the "wings" (aka the left and right cluster sets) of a prime end that was proved by Ursell and Young [UY] in 1951 and deserves to be far better known.

We will deduce Theorem 1.1 from Theorem 4.4 by developing a necessary and sufficient criterion for local connectivity at z_0 in terms of the prime ends of U. To state this result, we introduce the following notion.

1.2. Definition (Strong minimality)**.**
Let p be a prime end, and let z_0 belong to the impression of p. We say that z_0 is strongly minimal in p *if, for every sequence $w_j \in U$ converging to p which does not accumulate on z_0, there is a curve $\Gamma : [0, \infty) \to U$ that converges to p and passes through all w_j but does not accumulate on z_0 (as $t \to \infty$).*

This terminology is motivated by such a point being minimal with respect to Ursell and Young's ordering by priority; see Definition 4.2.

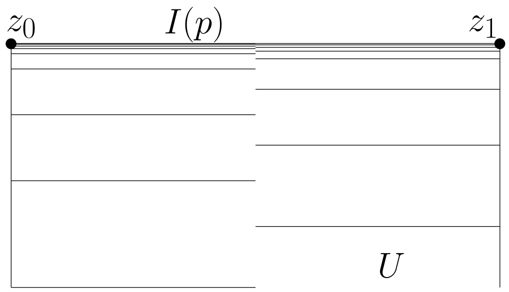
1.3. Theorem (Characterization of local connectivity)**.**
Let $z_0 \in \partial U$. Then K is locally connected at z_0 if and only if z_0 is strongly minimal in every prime end whose impression contains z_0.

The proof of Theorem 1.3 is elementary, and the result might almost be considered a restatement of the definition of local connectivity. However, it does provide an interesting and quite satisfying answer to our initial question about the connection between prime ends and local connectivity; in particular it contains Carathéodory's characterization of local connectivity of K (see Corollary 3.2). Theorem 1.1 follows from Theorem 1.3 and the aforementioned result by Ursell and Young (compare Corollary 4.5).

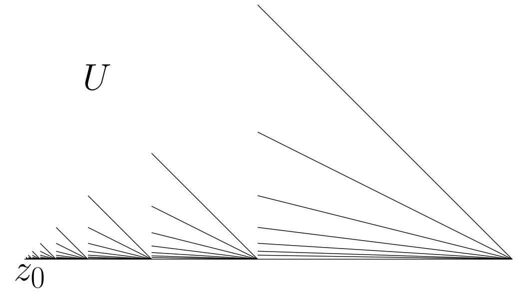
Basic notation. We denote the complex plane by $\mathbb{C}$, the Riemann sphere by $\hat{\mathbb{C}}$, and the unit disk by $\mathbb{D}$. We write $\mathbb{D}_\delta(z)$ for the (Euclidean) disk of radius δ around z.

Organization of the article. In Section 2, we define local connectivity at a point and discuss a number of variations of this definition. We also develop a simple characterization of local connectivity of K at z_0. Section 3 provides a short review of the theory of prime ends and the proof of Theorem 1.3. In Section 4, we discuss Theorem 4.4, by Ursell and Young, and deduce Theorem 1.1 from it. For completeness, we provide a proof of Theorem 4.4 in the Appendix.

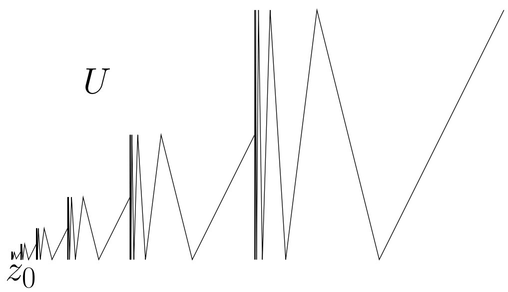
Acknowledgments. I had many interesting and enjoyable discussions on this subject over the years, in particular with Chris Bishop, David Epstein, Christian Pommerenke, Lex Oversteegen, Dierk Schleicher and the late Gerald Schmieder. I would like to thank Walter Bergweiler for a choice of seminar topic that not only introduced me to Pommerenke's excellent book [P], but also led me to discover Theorem 1.1 as an undergraduate at Kiel University in 1999. Finally, I am grateful to Christian Pommerenke and Lex Oversteegen for encouraging me to publish this note.



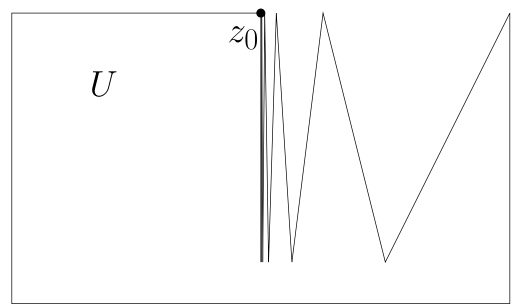
(a) The double comb



(b) The witch's broom



(c) Path connectivity



(d) K vs. ∂U

FIGURE 1. Several examples of simply connected domains and their boundaries. (a) illustrates Theorem 1.1: the interval at the top of the figure is the impression of a single prime end p, and $K = \hat{\mathbb{C}} \setminus U$ is locally connected at the two endpoints z_0 and z_1. Also note that these endpoints are *strongly minimal* in the sense of Definition 1.2, while the interior points are not. Examples (b) to (d) illustrate our remarks on the definition of local connectivity in Section 2.

2. Local connectivity

For the remainder of the paper, let $U \subset \hat{\mathbb{C}}$ be a simply connected domain whose complement $K := \hat{\mathbb{C}} \setminus U$ contains at least two points. We will assume without loss of generality that $\infty \in U$, so K is a compact, connected subset of the complex plane.

Recall that K is called *locally connected* if every point $z \in K$ has arbitrarily small connected (relative) neighborhoods in K. Following Milnor [M, Chapter 17], we say that K is locally connected *at a point* $z_0 \in K$ if z_0 has arbitrarily small connected neighborhoods in K. Sometimes this property is instead referred to as "connected im kleinen", and the term "locally connected at z_0" is instead reserved for what Milnor calls "openly locally connected", see below.

Observe that local connectivity of K at z_0 is equivalent to the condition that the connected component of $\{z \in K : |z - z_0| \leq \delta\}$ containing z_0 is a neighborhood of z_0 in K for all $\delta > 0$.

Remarks on the definition of local connectivity at a point. We note that there are many equivalent definitions of local connectivity of the entire space K which result in

different concepts when considered only near a point z_0. For example, we might say that K is *openly locally connected* at z_0 if the point has arbitrarily small *open* neighborhoods in K. It is well-known that K is locally connected if and only if it is openly locally connected at every point. However, if K is not locally connected as a whole, then it is quite possible that there are points $z_0 \in K$ where the set is locally connected but not openly locally connected. (A famous example is the "witch's broom"; see Figure 1(b).)

Similarly, local connectivity of K implies that K is locally arc-connected, but it is clearly possible for a compact space K to be locally connected at a point z_0, but not to contain *any* nontrivial curves passing through z_0. (See Figure 1(c).)

Finally, Carathéodory's theorem is often phrased as a statement on local connectivity of the *boundary* $\partial U = \partial K$, rather than of K. Indeed, these two are equivalent when we consider the entire space; however, local connectivity of ∂U at z_0 is a strictly stronger condition than local connectivity of K at z_0. (See Figure 1(d).)

We believe that, in our context, local connectivity of K at z_0 is the most natural among the possible concepts to consider. This is vindicated by the fact that we are able to obtain natural characterizations of this notion. Also, we should point out that our choice places the fewest restrictions on the point z_0, so that Theorem 1.1 takes its strongest form with this definition.

Separation theorems and preliminaries. We say that two points $z, w \in \hat{\mathbb{C}}$ are *separated* by a set K if they belong to different components of $\hat{\mathbb{C}} \setminus K$. Similarly, if $U \subset \hat{\mathbb{C}}$ is a domain, we sometimes say that z and w are *separated by K in U* if they belong to different components of $U \setminus K$.

We repeatedly use the following standard separation theorem [Ne, p. 110] due to Janiszewski: if K_1 and K_2 are compact subsets of the sphere whose intersection is connected, then a pair of points which is not separated by either of K_1 and K_2 is also not separated by the union $K_1 \cup K_2$.

We will also invoke the *boundary bumping theorem* [Na, Theorem 5.6]: if E is a subset of a compact, connected metric space K, then the boundary of every connected component of E intersects the boundary of E (in K).

Let us furthermore remind the reader that a *crosscut* C of a simply connected domain $U \subset \hat{\mathbb{C}}$ is a closed Jordan arc which intersects $\hat{\mathbb{C}} \setminus U$ exactly in its two endpoints. Every crosscut separates U into precisely two components. (Since U is homeomorphic to the complex plane, this is an immediate consequence of the Jordan Curve Theorem; compare [P, Proposition 2.12]. Observe that the argument applies more generally to any injective curve in U which accumulates at ∂U in both directions; we use this fact in the Appendix.)

Finally, we note the following simple result.

2.1. Lemma (Curves in a subdomain)**.**
Let $V \subset U$ be a domain, and let $z_0 \in \partial U$. Suppose that $\operatorname{dist}(z_0, \partial V \cap U) > \varepsilon$*.*

If $w_1, w_2 \in V$ can be connected by a curve $\gamma \subset U$ with $\operatorname{dist}(\gamma, z_0) > \varepsilon$*, then such a curve also exists in V.*

Proof. Let us set $A := \hat{\mathbb{C}} \setminus V \supset K$ and $B := K \cup \overline{\mathbb{D}_\varepsilon(z_0)}$. By assumption, neither A nor B separate w_1 and w_2. We claim that $X := A \cap B$ is connected.

Indeed, we have $X = K\cup((U\setminus V)\cap\overline{\mathbb{D}_\varepsilon(z_0)})$. Suppose, by contradiction, that there was a component L of X other than the one containing K; then in particular $z_0 \notin L$. Pick some boundary point w of L relative to $\overline{\mathbb{D}_\varepsilon(z_0)}$. Then $w \in \partial V$, but because $w \in L \subset U$ and $|w - z_0| \leq \varepsilon$, this contradicts our assumptions.

So X is connected. By Janiszewski's theorem, $A \cup B = (\mathbb{C} \setminus V) \cup \overline{\mathbb{D}_\varepsilon(z_0)}$ does not separate w_1 and w_2, as desired. ■

A characterization of local connectivity. Our proof of Theorem 1.3 (and, by extension, of Theorem 1.1) relies on the following necessary and sufficient condition for local connectivity of K at z_0. Compare Figure 2(c).

2.2. Proposition (Characterization of local connectivity)**.**
Let $z_0 \in \partial U$. Then K is locally connected at z_0 if and only if the following holds: for every $\delta > 0$, there is $\varepsilon > 0$ such that every point $w \in U \setminus \mathbb{D}_\delta(z_0)$ can be connected to ∞ by a curve $\gamma \subset U \setminus \overline{\mathbb{D}_\varepsilon(z_0)}$.

Proof. Suppose that K is locally connected at z_0, and let $\delta > 0$. Let L be the connected component of $K \cap \overline{\mathbb{D}_\delta(z_0)}$ containing z_0. Then L is a compact, connected neighborhood of z_0 in K. I.e., there is $\varepsilon > 0$ such that

$$\overline{\mathbb{D}_\varepsilon(z_0)} \cap K \subset L.$$

Let $w_1, w_2 \in U \setminus \mathbb{D}_\delta(z_0)$. Applying Janiszewski's theorem to K and $L \cup \overline{\mathbb{D}_\varepsilon(z_0)}$, we see that $K \cup \overline{\mathbb{D}_\varepsilon(z_0)}$ does not separate w_1 from w_2, as claimed.

For the converse direction, suppose that K is not locally connected at z_0. Then there is $\delta > 0$ such that the connected component L of $A := K \cap \overline{\mathbb{D}_\delta(z_0)}$ with $z_0 \in L$ is not a neighborhood of z_0 in K.

Let $\varepsilon > 0$. Then there is a point $z \in A \setminus L$ with $|z - z_0| \leq \varepsilon$. Write $A = A_0 \cup A_1$, where A_0 and A_1 are disjoint compact sets with $L \subset A_0$ and $z \in A_1$. By the boundary bumping theorem, both L and the component of A containing z intersect $\partial\mathbb{D}_\delta(z_0)$; in particular, they both intersect the circle $\partial\mathbb{D}_\varepsilon(z_0)$.

So we can pick an arc C of $\partial\mathbb{D}_\varepsilon(z_0) \setminus A$ which has one endpoint in A_0 and the other in A_1. Then $\overline{C}$ is a crosscut of U; let V be the component of $U \setminus C$ which does not contain ∞. We claim that V contains some point w with $|z_0 - w| \geq \delta$. Indeed, applying Janiszewski's theorem first to the sets A_0 and $\overline{C}$ and then to $A_0 \cup \overline{C}$ and A_1, we see that we can connect ∞ to any point $\tilde{w} \in V$ by a curve not intersecting $A \cup C$. If $|\tilde{w} - z_0| > \delta$, we set $w := \tilde{w}$. Otherwise let w be the last intersection point of this curve with $\partial\mathbb{D}_\delta(z_0)$; then $w \in V$.

By definition of V, any curve γ connecting ∞ to w must intersect C, and hence have $\operatorname{dist}(\gamma, z_0) \leq \varepsilon$, as required. ■

3. Prime ends

We refer the reader to [M, Chapter 17] for an excellent short treatment of the theory of prime ends, and to [P, Chapter 2] for further results. Here, we will only introduce the basic definitions, and give no proofs. As before, U is a simply connected domain whose complement K omits more than one point, and $\infty \in U$ for simplicity.

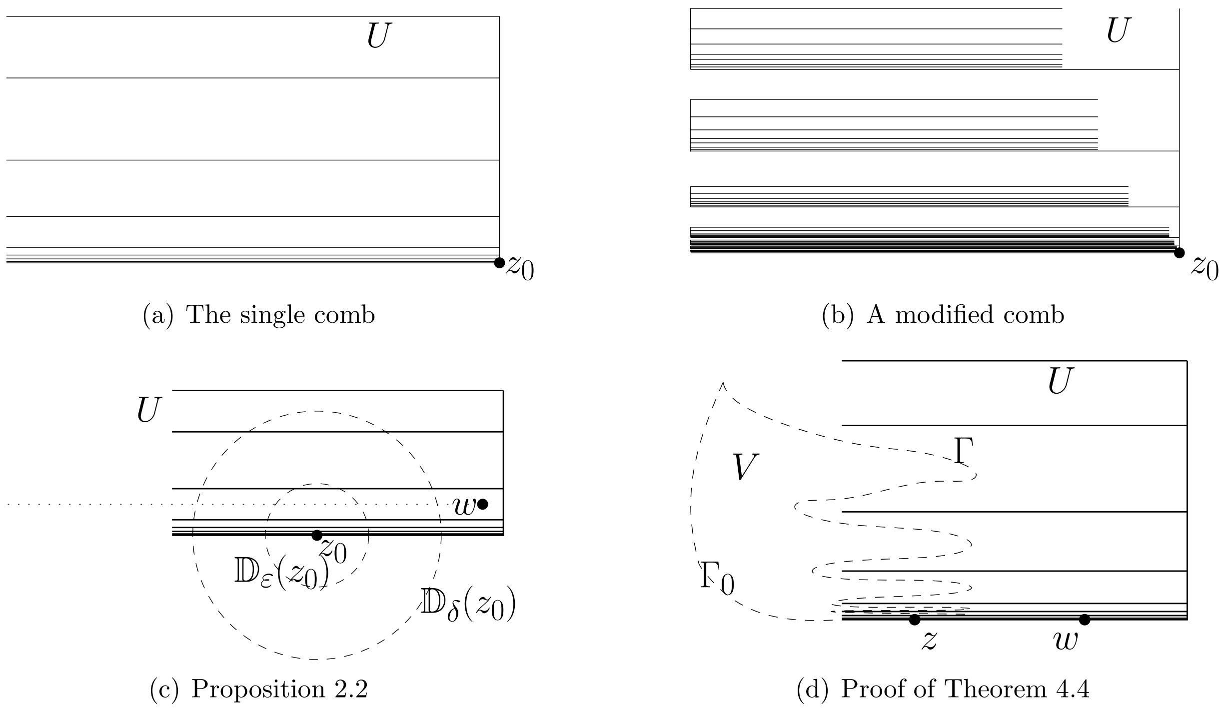


FIGURE 2. The single comb and variations. In (a), the set $K = \hat{\mathbb{C}} \setminus U$ is locally connected at z_0 (but not in any other point of the interval at the bottom edge of the picture). In contrast, the point z_0 is not a point of local connectivity in (b): this shows that the structure of a prime end's impression, and the order of priority from Definition 4.2, is not sufficient to detect local connectivity. Figure (c) illustrates the statement of Proposition 2.2: K is not locally connected at z_0, and any curve connecting w to infinity must intersect the disk $\mathbb{D}_\varepsilon(z_0)$. Finally, (d) indicates the setup in the proof of Theorem 4.4: the curve Γ accumulates on z but not on w, while Γ_0 accumulates on the set $\Pi(p)$ of principal points. Together with their accumulation sets, they separate w from the region V.

If C is a crosscut of U with $\infty \notin C$, then $U \setminus C$ has exactly one component which does not contain ∞; let us denote this component by U_C. A *prime end* of U is represented by a sequence of pairwise disjoint crosscuts (C_n) which satisfy $\operatorname{diam} C_n \to 0$ and $C_{n+1} \subset \overline{U_{C_n}}$. Two such sequences (C_n) and $(\tilde{C}_n)$ represent the same prime end if $\tilde{C}_j \subset U_{C_n}$ for all n and all sufficiently large j, and vice versa.

The *impression* of p is defined as

$$I(p) := \bigcap_n \overline{U_{C_n}} \subset \partial U;$$

an impression is *trivial* if it consists of a single point. The set of *principal points*, $\Pi(p) \subset I(p)$, consists of those points which are accumulated on by some sequence of crosscuts representing p.

There is a natural way to define a topology on

$$\breve{U} := U \cup \{p : p \text{ is a prime end of } U\}$$

such that a sequence $w_j \in U$ converges to p if and only if $w_j \in U_{C_n}$ for all n and all sufficiently large j. With this topology, $\breve{U}$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$. In fact, if $\varphi : \mathbb{D} \to U$ is a conformal isomorphism, then φ extends continuously to a homeomorphism $\varphi : \overline{\mathbb{D}} \to \breve{U}$.

We say that a curve $\Gamma : [0, \infty) \to U$ *converges to* p if $\Gamma(t) \to p$ in the topology of $\breve{U}$. Note that the set of accumulation points of such a curve Γ necessarily contains $\Pi(p)$.

Proof of Theorem 1.3. Suppose that K is locally connected at z_0, and consider a prime end p with $z_0 \in I(p)$. Let (w_j) be a sequence as in the definition of strong minimality (Definition 1.2), and set $\delta := \inf |z_0 - w_j|$. By Proposition 2.2, there is $\varepsilon_0 > 0$ such that each w_j can be connected to ∞ by a curve γ_j with $\operatorname{dist}(\gamma_j, z_0) > \varepsilon_0$.

Let (C_n) be a sequence of crosscuts representing p, and set $U_n := U_{C_n}$. Without loss of generality, we may assume that no C_n has z_0 as an endpoint (otherwise, we simply remove this crosscut from the sequence).

For sufficiently large j, we have $w_j \in U_n$, and hence $\gamma_j \cap C_n \neq \emptyset$. Since $\operatorname{dist}(\gamma_j, z_0) > \varepsilon_0$ and $\operatorname{diam}(C_n) \to 0$, we see that

$$\varepsilon_1 := \inf_n \operatorname{dist}(z_0, C_n) > 0;$$

we set $\varepsilon := \min(\varepsilon_0, \varepsilon_1)$.

By construction, we can connect w_j and w_{j+1} by a curve Γ_j which does not intersect $\overline{\mathbb{D}_\varepsilon(z_0)}$ (e.g., $\Gamma_j = \gamma_j \cup \gamma_{j+1}$). If w_j and w_{j+1} both belong to the same U_n, we can, by Lemma 2.1, furthermore choose Γ_j such that $\Gamma_j \subset U_n$. The curve $\Gamma := \bigcup \Gamma_j$ converges to the prime end p, contains all points w_j, and does not intersect $\overline{\mathbb{D}_\varepsilon(z_0)}$.

Now suppose, conversely, that K is not locally connected at z_0. Then, by Proposition 2.2, there is a constant $\delta > 0$ with the following property: for every $n \in \mathbb{N}$ there is a point $\omega_n \in U$ with $|\omega_n - z_0| \geq \delta$ such that any curve connecting ω_n to ∞ within U must pass within distance at most $1/n$ of z_0.

Clearly we have $\omega_n \to \partial U$. Let p be some accumulation point of ω_n in the prime end compactification $\breve{U}$ of U. We define a sequence $(w_j)_{j \geq 0}$ by setting $w_0 := \infty$ and $w_j := \omega_{n_j}$, where $(\omega_{n_j})_{j \geq 1}$ is a subsequence converging to p in $\breve{U}$.

Then any curve Γ tending to p and containing all points w_j will, in particular, connect $w_0 = \infty$ to $w_j = \omega_{n_j}$. Hence $\operatorname{dist}(\Gamma, z_0) = 0$ by choice of (ω_n), which implies that $z_0 \in I(p)$ and that z_0 is not strongly minimal in p. ∎

3.1. Corollary (Local connectivity at principal points)**.**
Suppose that K is locally connected at a principal point z_0 of p. Then $I(p) = \{z_0\}$.

Proof. We prove the contrapositive. Suppose that $I(p) \neq \{z_0\}$; then we can pick some $z \in I(p) \setminus \{z_0\}$. Take a sequence of points $w_k \in U$ converging to z such that w_j converges to p in the prime end topology. Since z_0 is a principal point, *any* curve converging to p (containing all w_j or not) must accumulate on z_0, and hence z_0 is not strongly minimal. By Theorem 1.3, K is not locally connected at z_0. ∎

Remark. This fact is easy to prove also without Theorem 1.3: Let (C_n) be a sequence of crosscuts converging to z_0. Pick a closed connected neighborhood Z_ε of z_0 in K satisfying $Z_\varepsilon \subset \mathbb{D}_\varepsilon(z_0)$; we may suppose without loss of generality that $\hat{\mathbb{C}} \setminus Z_\varepsilon$ is connected. Then, for sufficiently large n, the curve C_n is a crosscut of $\hat{\mathbb{C}} \setminus Z_\varepsilon$. It follows that U_n, and hence $I(p)$, is contained in $\overline{\mathbb{D}_\varepsilon(z_0)}$. Since ε was arbitrary, the claim follows.

3.2. Corollary (Carathéodory's theorem)**.**
The set K is locally connected if and only if every prime end impression is trivial.

Proof. Since K is locally connected at every point of its interior, we only need to consider local connectivity at points of ∂U.

If K is *not* locally connected at some point $z_0 \in \partial U$, then by Theorem 1.3 there is some prime end p with $z_0 \in I(p)$ such that z_0 is not strongly minimal in p. In particular, there is some sequence (w_j) converging to p but not converging to z_0, so $I(p)$ is not trivial.

On the other hand, if there is some nontrivial prime end impression $I(p)$, then K is not locally connected at any point of $\Pi(p)$ by the previous corollary. ■

4. The left and right wings of a prime end

The article [UY] studied the *left and right wings* of a prime end. Today these are more commonly referred to as the *left and right cluster sets*, or as the *one-sided impressions* [CP2]. We prefer to use "wing" here, as it seems to be the original term used when these sets are investigated as topological objects related to the domain U and its boundary, rather than as an aspect of a conformal mapping $\varphi : \mathbb{D} \to U$. For simplicity, we will nonetheless not give a purely topological definition, but rather use the conformal mapping φ.

Let p be a prime end of U. We say that a curve $\Gamma : [0, \infty) \to U$ converges to p *from the left* if Γ converges to p, and furthermore

$$\operatorname{Im}\left(\frac{\varphi^{-1}(\Gamma(t))}{p}\right) \geq 0$$

for all sufficiently large t. (Here we again identify the prime end p with the corresponding point $p = e^{2\pi i\vartheta}$ on the unit circle.) We say that any accumulation point $z_0 \in I(p)$ of such a curve Γ belongs to the *left wing* of p, and write $I^+(p)$ for all such points.

The *right wing* $I^-(p)$ is defined analogously. We note that $I(p) = I^+(p) \cup I^-(p)$ and $\Pi(p) \subset I^-(p) \cap I^+(p)$.

4.1. Example.
In Figure 1(a), the interval at the top of the picture is the only nontrivial prime end impression. The midpoint m of this interval is the unique principal point; the left and right wings are the intervals $[z_0, m]$ and $[m, z_1]$, respectively.

In Figures 2(a) and 2(b), the prime end at the bottom of the picture has a trivial right wing, containing only the unique principal point, while the left wing consists of the entire interval at the bottom of the picture.

The prime end p is called *symmetric* if $I^-(p) = I^+(p)$. By the Collingwood Symmetry Theorem [P, Proposition 2.21], all but countably many prime ends are symmetric. Compare [CP1] for interesting results on symmetric prime ends (among other things).

4.2. Definition (Priority)**.**
Let $z, w \in I^-(p)$. We say that z has priority over *w (in $I^-(p)$) if every curve Γ which converges to p from the left and accumulates on w must also accumulate on z.*
(Priority in $I^+(p)$ is defined analogously.)

4.3. Example.
In the left wing of the nontrivial prime end expression of Figure 2(a), the order of priority coincides with horizontal order: if z, w belong to the interval at the bottom of the picture and z is to the left of w, then z has priority over w. The same is true in Figure 2(b).

4.4. Theorem (Priority is a total relation [UY])**.**
Let z, w belong to the same wing of the prime end p. Then z has priority over w or w has priority over z.

Now suppose that $z_0 \in I(p)$ is strongly minimal in p. Then z_0 is minimal with respect to priority; indeed, z_0 cannot have priority over any point in either wing. Hence Theorem 4.4 implies:

4.5. Corollary (At most two strongly minimal points)**.**
Each wing of the prime end p contains at most one point which is strongly minimal in p. ■

Proof of Theorem 1.1. We just proved that $I(p)$ contains at most two points which are strongly minimal (at most one for each wing). We also know by Theorem 1.3 that local connectivity of K at z_0 requires strong minimality of z_0 in p. This proves the theorem.

If the prime end is symmetric, then both wings are equal and Corollary 4.5 implies that $I(p)$ contains at most one strongly minimal point, establishing the remark after the statement of Theorem 1.1. ■

Remark. One might ask whether strong minimality can be expressed solely in terms of the order of priority on $I(p)$; this is not the case. Indeed, the first two examples in Figure 2 both have a prime end whose impression is the bottom interval of the picture, and as discussed above the corresponding orders of priority coincide. However, in the first figure z_0 is strongly minimal, while in the second figure it is not.

Appendix: Proof of the Ursell-Young Theorem

Proof of Theorem 4.4. Suppose that z and w both belong to (say) the left wing of the prime end p, and that w does not have priority over z. That is, there is a curve $\Gamma : [0, \infty) \to U$ converging to p from the left whose accumulation set A contains z but not w. For simplicity, let us assume that Γ is injective. (It is not hard to see that we can always find an injective curve with the same accumulation set. Alternatively, with minor modifications the proof will also apply in the general case.) We need to show that z has priority over w. We may assume that $z \notin \Pi(p)$, as every principal point has priority over all other points by definition.

Let $\Gamma_0 : [0, \infty) \to U$ be the "central curve"
$$\Gamma_0(t) := \varphi(e^{-1/t} \cdot p)$$
separating the left and right wings of p. The limit set of Γ_0 is the set of *radial limit points* of φ at p, which is well-known to consist exactly of $\Pi(p)$ [P, Theorem 2.16]. We may assume without loss of generality that $\Gamma(0) = \Gamma_0(0)$ and that $\Gamma\big((0, \infty)\big) \cap \Gamma_0 = \emptyset$ (recall that Γ converges to p from the left).

Since Γ_0 and Γ_1 accumulate only on ∂U, the set $U \setminus (\Gamma_0 \cup \Gamma)$ has exactly two components. Since Γ_0 and Γ both converge to the same prime end p, one of these components, call it V, accumulates on no other prime ends in the topology of $\breve{U}$. (Compare Figure 2(d).) We pick some arbitrary base point $x \in V$.

Claim 1. We have $\partial V = \Gamma_0 \cup \Gamma \cup A =: F$. In particular, F separates x from w.

Proof. Let $\tilde{U}$ be the component of $\hat{\mathbb{C}} \setminus A$ containing U and let $\tilde{V}$ be the component of $\tilde{U} \setminus (\Gamma_0 \cup \Gamma)$ containing V. Then $V = \tilde{V} \cap U$; we need to show that $V = \tilde{V}$.

Indeed, otherwise $\tilde{V}$ is a neighborhood of some point of ∂U; in particular $\tilde{V}$ and hence V contains some crosscut of U. However, this is a contradiction to the choice of V, since every crosscut accumulates on two distinct prime ends of U. △

Claim 2. Let ε be sufficiently small. Let $T > 0$ be minimal with $|\Gamma(T) - z| \le \varepsilon$. Then the set
$$K_1(\varepsilon) := \overline{\mathbb{D}_\varepsilon(z)} \cup \Gamma\big([T, \infty)\big) \cup A$$
does not separate x and w (in $\hat{\mathbb{C}}$).

Proof. Let $w' \in U$ with $|w - w'| < \operatorname{dist}(w, A)$. By connecting x to w' in U and w' to w by a straight line segment, we obtain a curve $\alpha \subset \mathbb{C} \setminus A$ which connects x to w. Set $\delta := \operatorname{dist}(\alpha, A)$. If ε is sufficiently small, then $K_1(\varepsilon)$ is contained in a δ-neighborhood of A, and hence does not intersect α. △

Let (C_n) be a sequence of crosscuts representing p; we may choose these so that $C_n \cap \Gamma_0$ consists of a single point for every n. Let U_n be the component of $U \setminus (C_n \cup \Gamma_0)$ which contains $\Gamma(t)$ for large t. Then a curve converges to p from the left if and only if it is eventually contained in every U_n.

Claim 3. Let $\varepsilon > 0$. Then there are n_0 and δ with the following property. If $n \ge n_0$ and $w' \in U_n$ with $|w - w'| < \delta$, then any curve in U_n connecting w' to C_n intersects $\overline{\mathbb{D}_\varepsilon(z)}$.

Proof. By decreasing ε, if necessary, we may assume that x can be connected to every crosscut C_n by a curve $\beta_n \subset V$ which does not intersect $\overline{\mathbb{D}_\varepsilon(z)}$. This is possible because Γ_0 does not accumulate on z.

By Claim 2, we may also assume that $K_1 = K_1(\varepsilon)$ does not separate x and w. Consider the set
$$K_2 := \overline{\mathbb{D}_\varepsilon(z)} \cup A \cup \Gamma_0 \cup \Gamma\big([0, T]\big).$$
Observe that $K_1 \cap K_2 = \mathbb{D}_\varepsilon(z) \cup A$ and $F \subset K_1 \cup K_2$. Hence it follows from Claim 1 and Janiszewski's theorem that K_2 separates x and w.

Choose δ sufficiently small that $\mathbb{D}_\delta(w) \cap K_2 = \emptyset$, and choose n_0 such that $C_n \cap \overline{\mathbb{D}_\varepsilon(z)} = \emptyset$ and $U_n \cap \Gamma\big([0, T]\big) = \emptyset$ for $n \ge n_0$.

If $w' \in U_n$ with $n \geq n_0$ and $|w' - w| < \delta$, then K_2 separates w' and x. Let $\gamma \subset U_n$ be a curve connecting w' to C_n. Combining γ with a piece of C_n and the curve β_n, we obtain a curve in $U \setminus (\Gamma_0 \cup \Gamma([0, T]))$ connecting w' to x. This curve must intersect K_2, and hence $\overline{\mathbb{D}_\varepsilon(z)}$. Since C_n and β_n do not intersect this disk, it follows that γ intersects $\overline{\mathbb{D}_\varepsilon(z)}$, as claimed. $\triangle$

The proof of Theorem 4.4 is now complete: suppose that $\tilde{\Gamma}$ is a curve converging to p from the left and accumulating on w, and let $\varepsilon > 0$. Let δ and n_0 be as in Claim 3, and pick $n \geq n_0$ sufficiently large so that $\tilde{\Gamma} \not\subset U_n$. Since $\tilde{\Gamma}$ contains some point $w' \in U_n$ with $|w' - w| < \delta$, it follows that $\tilde{\Gamma}$ intersects $\overline{\mathbb{D}_\varepsilon(z)}$. ■

References

[C] Constantin Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Math. Ann. **73** (1913), 323–370.

[CP1] J. J. Carmona and C. Pommerenke, *On prime ends and plane continua*, J. London Math. Soc. (2) **66** (2002), no. 3, 641–650.

[CP2] Joan J. Carmona and Christian Pommerenke, *Decomposition of continua and prime ends*, Comput. Methods Funct. Theory **3** (2003), no. 1-2, 253–272.

[H] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, pp. 467–511.

[K] Jan Kiwi, *$\mathbb{R}$eal laminations and the topological dynamics of complex polynomials*, Adv. Math. **184** (2004), no. 2, 207–267.

[M] John Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[Na] Sam B. Nadler, Jr., *Continuum theory. An introduction*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker Inc., New York, 1992.

[Ne] M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge, At the University Press, 1951, 2nd ed.

[P] Christian Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften, vol. 299, Springer-Verlag, Berlin, 1992.

[UY] H. D. Ursell and L. C. Young, *Remarks on the theory of prime ends*, Mem. Amer. Math. Soc., **3** (1951), 29.

Department of Mathematical Sciences, University of Liverpool, Liverpool, United Kingdom L69 7ZL

E-mail address: `l.rempe@liverpool.ac.uk`